THE CONVERGENCE OF E – SOLUTIONS IN TERMS OF EPIGRAPHICAL DISTANCE

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ABSTRACT

The purpose of this paper is to investigate the set of $\varepsilon-$ optimal solutions to optimization problems from a metrical point of view, and generalize some results in literature that are dealing from a topological point of view. Precisely, we show that the sequence (f_n) is epigraphical distance convergent to f if and only if for each $\varepsilon > 0$, the sequence of sets $(\varepsilon-\arg\min f_n)$ is epigraphical distance convergent to $(\varepsilon-\arg\min f)$. An analogous result holds for $\varepsilon-$*subdifferentials* of convex lower semi-continuous functions defined on a Banch space and also for $\varepsilon-$*projctions* of a point to convex closed subset in X .

KEYWORDS: Epigraphe, Epigraphical Distance, Hausdorff Distance, Soulotion, Lower Semicontinuous Functions, Optimition.

INTRODUCTION

Let us considered the minimization problem of the form:

$$P_f: \qquad \inf\{f(x), x \in X\}$$

where X is a real normed space and $f : R \to \overline{R}$ a real extension function . We recall that under some condition there exists a solution of problem P_f (example, when f is coercive i.e f has bounded level set) , but in general the solutions are not existed to this problem and the set of optimal solutions $(\arg\min f)$ may be empty. It is important to use the notion of $\varepsilon-$*solutions* that is considered as more natural from a numerical point of view. However, a major advantage of the set $(\varepsilon-\arg\min f)$ is allows non void, refers to the systematic studies in [1,4,14,16,20].

We suppose that the problem P_f takes the formula:

$$P_{f_n} \ : \qquad \inf\{f_n(x), x \in X\}$$

A natural question arises : If the sequence (f_n) converges to f , how does the set of solutions $(\varepsilon-$*solutions*$)$ to problem P_{f_n} converge to the set of solutions$(\varepsilon-$*solutions*$)$ to problem P_f ? This problem has been studied a topological point of view by many lectures, $[2,14,21,16]$ .Most of the stability results are topological in nature and it is known that the good notion of topological which yields stability for minimization problem is the notion epi-convergence, see $[2,19,20,21]$ for example.

The main propose of this article is to discuss this question and another more general questions from a metrical point of view, in using the notion of ρ-Hausdorff distance (epigraphical-distance) introduced in $[5]$. This paper is organized as follows. In section 1, we fix the notations and recall some definitions and some known results. In section 2, we give the premier main result Theorem 2.1concerned the level sets that generalized many results in $[24, Theorem 1]$, $[23, p.199]$, , $[20, Theorem 2.1]$ .We show that the sequence (f_n) is epigraphcal-distance convergent to f if and only if for each $\lambda > \inf f$, the level sets of the f_n at height λ is ρ-Hausdorff distance convergent to the level set of the f at height λ. Moreover, if we take $(\lambda_n \to \lambda \ in\ R)$ the result stays hold. In Theorem 2.1, we give the second main result concerned the$(\varepsilon - solutions)$, we show that the sequence (f_n) is epigraphcal-distance convergent to f if and only if for each $\varepsilon > 0$ the sets of $(\varepsilon - \arg\min f_n)$ are ρ-Hausdorff distance convergent to $(\varepsilon - \arg\min f)$. As by product, we obtain the stability of the $\varepsilon - subdifferentials$ of convex lower semicontinuous function defined on a Banch space. Finally, we study the $\varepsilon - projctions$ of a point on a convex set in X.

1. Notations and Definitions

Let $(X, \|.\|)$ be a normed linear space and $(X^*, \|.\|_*)$ its dual, $[7,8,9,10]$the duality pairing between $x^* \in X^*$ and $x \in X$ is denoted by $\langle x, x^* \rangle$, and let $f : R \to R \cup \{-\infty, +\infty\}$ of the real valued extension function defined on X, we well denote the set of the real valued extended functions defined on X by $\overline{R}^X$. For a function $f \in \overline{R}^X$ the set :

$$epi\, f = \{ (x, \lambda) \in X \times R \ / \ f(x) \leq \lambda \}$$

is called the epigraph of f , and f is called convex (lower semiconti-nuous) if its epigraph is a convex (closed) subset of $X \times R$. Furthermore, f is called proper if its epigraph nonempty, or if its domain is nonempty;

$$dom f = \{x \in X \ / \ f(x) < +\infty\} \neq \phi$$

It may be observed that the projection of $epi f$ on X is the effective domain of f.

Again, $\Gamma(X)$ well denote the proper, lower semi continuous convex functions defined on X, and dually, $\Gamma^*(X^*)$ well denote the proper, weak* lower semi continuous convex functions defined on X^*. It is well known to each nonempty closed convex subset C of X its indicator function $\delta(.,C) \in \Gamma(X)$, defined by the formula

$$\delta(.,C) = \begin{cases} 0 & if\ x \in C \\ +\infty & if\ x \notin C \end{cases} \quad (1)$$

- For $f \in \Gamma(X)$, its *conjugate* $f^* \in \Gamma^*(X^*)$ is defined by the familiar formula :

$$f^*(x^*) = \sup_{x \in X}\{<x, x^*> - f(x)\}, \forall x^* \in X^*$$

- For $\varepsilon > 0$ and $f \in \overline{R}^X$, the $\varepsilon -$ $\arg\min f$ is defined by :

$$\varepsilon - \arg\min f = \left\{u \in X \ / f(u) \leq \sup\left\{\inf_{x \in X} f(x) + \varepsilon, -\frac{1}{\varepsilon}\right\}\right\} \quad (2)$$

This definition is justified because it is possible for the quantity $\inf f(x)$ to be equal to $-\infty$, and from a numerical point of view, this notion is quit natural. Moreover, the set $\varepsilon - \arg\min f$ is nonempty and reduces to the exact $\arg\min f$ where $\varepsilon = 0$.

- For $\varepsilon > 0$, the $\varepsilon - subdifferential$ of $f \in \overline{R}^X$ at x_0, denoted by $\partial_\varepsilon f(x_0)$, is defined by :

$$\partial_\varepsilon f(x_0) = \{x^* \in X^* / \ f(x) \geq f(x_0) + <x - x_o, x^*> - \varepsilon \ ; \forall x \in X\}$$

$$= \{x^* \in X^* / \ f(x) + f^*(x^*) - <x, x^*> \leq \varepsilon\} \quad (3)$$

This set is convex (closed) if f is convex (lower semi continuous), which reduces to the exact sub differential where $\varepsilon = 0$.

- For $f \in \overline{R}^X$, and for each real λ, the level set at height λ of f, is defined by :

$$S_\lambda^f := \{x \in X \ / f(x) \leq \lambda\}$$

The function f is lower semi continuous if S_λ^f is closed subset in X.

- For every $\varepsilon > 0$, the ε - *project* of x_0 to C, denoted by ε - $proj\,(x_0, C)$ is defined by :

$$\varepsilon - proj\,(x_0 , C) = \left\{ x \in C \ / \ \| x - x_0 \| \leq d(x_0 , C) + \varepsilon \right\}$$

where

$$d(x_0 , C) = \inf |x_0 - x| \ , \ \left(if \ C = \phi \ , \ d(x_0 , C) = +\infty \right)$$

- ($\rho - Housdoroff \ distances$ on X) $[3]$

For each $\rho \geq 0$, ρB denotes the closed ball of radius ρ ; and for any subset C of X , we define C_ρ by : $C_\rho := C \cap \rho B$

For any pair C and D of subsets of X , the *Housdoroff excess* of C over D is defined by :

$$haus_\rho(D , C) = Sup\left\{ e(C_\rho , D) \ ; \ e(D_\rho , C) \right\}$$

$$e(C , D) = \sup_{x \in C} d(x , D) \ ; \ \left(e(C , D) = 0 \ if \ C = \phi \right)$$

and for all $\rho \geq 0$, the $\rho - Housdoroff \ distances$ between C and D is defined by :

$$haus_\rho(D , C) = Sup\left\{ e(C_\rho , D) \ ; \ e(D_\rho , C) \right\}$$

A sequence of sub sets $(D_n)_{n \in N}$ of X , is said to converge with respect to the $\rho - Housdoroff \ distances$ to some D iff for all $\rho \geq 0$,

$$\lim_{n \to \infty} haus_\rho(D_n , D) = 0 \qquad (4)$$

- ($\rho - Housdoroff \ distances$ on $\overline{R}^X$) $[3]$

a) For all $\rho \geq 0$, the $\rho - Housdoroff \ distances$ between two functions $f , g \in \overline{R}^X$ is defined by :

$$H_\rho(f , g) = haus_\rho(epi \ f \ , \ epi \ g)$$

where $epi \ f \ and \ epi \ g$ are two subsets of $X \times R$, and the ball of $X \times R$ is the set :

$$\rho B_{X \times R} = \left\{ (x , \alpha) \in X \times R \ / \ \| x \| \leq \rho \ , \ |\alpha| \leq \rho \right\}$$

b) A sequence of functions $(f_n)_{n \in N}$ of $\overline{R}^X$, is said to converge with respect to the $\rho - Housdoroff \ distances$ to some f iff for all $\rho \geq 0$,

$$\lim_{n \to \infty} H_\rho(f_n , f) = 0 \qquad (5)$$

The concept $\rho - Housdoroff \ distances$ is also called the ρ- *epigraphical distance* introduced in $[3,5]$ and has been developed by many authors in various field $[7,8,9,10]$.

We recall two fundamental results, the first gives the bicontinuity between the functions in

$\Gamma(X)$ and its conjugates of $\Gamma^*(X^*)$, and the second gives the continuity of sum the functions in $\Gamma(X)$, with respect to the $\rho-$Housdorff distance .

Proposition 1.1 [5]

Let $\{ f_n , f ; n \in N \}$ be a sequence of functions in $\Gamma(X)$. Then for all $\rho \geq 0$, the following statements are equivalent :

(i) $\lim_{n\to\infty} H_\rho (f_n , f) = 0$

(ii) $\lim_{n\to\infty} H_\rho (f_n^*, f^*) = 0$.

Proposition 1.2 [8]:

Let $\{ f_n , f ; n \in N \}$ be a sequence of functions in $\Gamma(X)$, and let $g \in \Gamma(X)$ with g is finite valued and continuous function on X. Then for all $\rho \geq 0$,

$$\lim_{n\to\infty} H_\rho (f_n , f) = 0$$

implie

$$\lim_{n\to\infty} H_\rho (f_n + g , f + g) = 0 \ .$$

2. The main results .

The level sets are investigated by many authors from a topological point of view, see [2 ,20 ,22, 23].We study the level sets from the metric point of view, and give a characterization of the ρ- epigraphical distance in term of convergence of level sets.

Theorem 2.1

Let $\{ f_n , f ; n \in N \}$ be a sequence of functions in $\overline{R}^X$, let $\lambda \in R$ such that $\lambda > \inf f$, for any $\rho \geq 0$ we have:

1) If $\lim_{n\to\infty} H_\rho (f_n , f) = 0$, then $\lim_{n\to\infty} haus_\rho (S_\lambda^{f^n} , S_\lambda^f) = 0$

2) If $\inf f_n > \inf f$, and $\lim_{n\to\infty} haus_\rho (S_\lambda^{f^n}, S_\lambda^f) = 0$,then $\lim_{n\to\infty} H_\rho (f_n ,f) = 0$.

Proof :

Proof 1) It is sufficient to prove that: for each $\rho \geq 0$,

$$e\left[(S_\lambda^{f_n})_\rho , S_\lambda^f \right] \xrightarrow[n\to\infty]{} 0$$

Let $x \in (S_\lambda^{f_n})_\rho$, then for each $n \in N$

$$f_n (x) \leq \lambda \quad \textit{and} \quad \| x \| \leq \rho \qquad (6)$$

Hence

$$(x , \lambda) \in (epi\, f_n)_{\rho_1}$$

where

$$\rho_1 = Max\ \{\ \rho, |\lambda|\}$$

By definition (5) of H_ρ, for each $\varepsilon > 0$, there exist $(\xi_\varepsilon, \lambda_\varepsilon) \in epi\, f$ such that ,

$$\| x - \xi_\varepsilon \| \leq H_{\rho_1}(f_n, f\ ) + \varepsilon \qquad (7)$$

$$| \lambda - \lambda_\varepsilon | \leq H_{\rho_1}(f_n, f\ ) + \varepsilon \qquad (8\ )$$

From (8) , it follows that ;

$$\lambda_\varepsilon \leq \lambda + H_{\rho_1}(f_n, f\ ) + \varepsilon$$

and since

$$f\ (\xi_\varepsilon) \leq \lambda_\varepsilon$$

we derive

$$f\ (\xi_\varepsilon) \leq \lambda + H_{\rho_1}(f_n, f\ ) + \varepsilon \qquad (9)$$

Passing to the limit of the inequality (9) , as $n \to \infty$ and taking $(\varepsilon \to 0)$, and using the assumption , we obtain $f\ (\xi_\varepsilon) \leq \lambda$ that is,

$$\xi_\varepsilon \in S_\lambda^f \qquad (10)$$

On the other hand, from (7) as $n \to \infty$, we deduce

$$\| x - \xi_\varepsilon \| \leq \varepsilon \qquad (11)$$

It follows from (10) and (11) that

$$d\,(x\ , S_\lambda^f\ ) \leq \varepsilon$$

This conclusion is true for each $x \in (S_\lambda^{f_n})_\rho\,; n \in N$. Thus, we conclude that

$$e\left[(S_\lambda^{f_n})_\rho\ , S_\lambda^f\right] \xrightarrow[n \to \infty]{} 0$$

Finally, we can prove in the same way that $e\left[(S_\lambda^f)_\rho\ , S_\lambda^{f_n}\right] \xrightarrow[n \to \infty]{} 0$, i.e

$$\lim_{n \to \infty} haus_\rho(\ S_\lambda^{f^n}, S_\lambda^f\ ) = 0$$

***Proof* 2)** : Let $(x\, , \eta\ ) \in (epi\, f_n)_\rho$, then for each $n \in N$

$$f_n(x\ ) \leq \eta\ \ ,\ \ \| x \| \leq \rho\ \ and\ \ |\eta| \leq \rho$$

hence $x \in (S_\eta^{f_n})_\rho$.

By definition (4) of $haus_\rho$, for each $\varepsilon > 0$, there exist $\xi_\varepsilon \in S_\eta^f$ such that ,

$$\| x - \xi_\varepsilon \| \leq haus_\rho(\ S_\eta^{f_n}, S_\eta^f\ ) + \varepsilon \qquad (12)$$

Since $\inf f_n > \inf f$, we have $\eta \geq f_n(x\ ) \geq \inf f_n > \inf f$, hence $\eta > \inf f$ and so $\lim_{n \to \infty} haus_\rho(S_\eta^{f^n}, S_\eta^f\ ) = 0$, we derive from (12) that

$$\| x - \xi_\varepsilon \| \leq \varepsilon \qquad (13)$$

On the other hand, for every $\varepsilon > 0$ there always exist $\eta_\varepsilon \in R$ such that

$$|\eta - \eta_\varepsilon| \leq \varepsilon \qquad (14)$$

It follows from (14) that , $\eta \leq \eta_\varepsilon$ and since $\xi_\varepsilon \in S_\eta^f$ thus $f\ (\zeta_\varepsilon) \leq \eta_\varepsilon$, this yields

$$(\xi_\varepsilon, \eta_\varepsilon) \in epi\, f \qquad (15)$$

It follows from (13) and (15) that

$$d\left[(x\ ,\eta)\, , epi\, f\ )\right] \leq \varepsilon$$

This conclusion is true for each $(x,\eta) \in (epi\, f_n)_\rho$. Thus we conclude that

$$e\left[(epi\, f_n)_\rho\, , epi\, f\right] \xrightarrow[n\to\infty]{} 0$$

In the same manner, we can prove that $e\left[(epi\, f)_n\, , epi\, f_n\right] \xrightarrow[n\to\infty]{} 0$ i.e

$$\lim_{n\to\infty} H_\rho(f_n, f) = 0 \ . \ \blacksquare$$

Proposition 2.2

Under the assumption of Theorem 2.1 and $\left(\forall\, \lambda_n \xrightarrow[n\to\infty]{} \lambda \ in\ R\right)$, we have :

1) If $\lim_{n\to\infty} H_\rho(f_n, f) = 0$, then $\lim_{n\to\infty} haus_\rho(S_{\lambda_n}^{f^n}, S_\lambda^f) = 0$

2) If $\inf f_n > \inf f$, and $\lim_{n\to\infty} haus_\rho(S_{\lambda_n}^{f^n}, S_\lambda^f) = 0$, then $\lim_{n\to\infty} H_\rho(f_n, f) = 0$.

Proof:

The proof is similar to the proof of Theorem 2.1 It is sufficient to prove , for example, that for each $\rho \geq 0$,

$$e\left[(S_{\lambda_n}^{f_n})_\rho\, , S_\lambda^f\right] \xrightarrow[n\to\infty]{} 0$$

Let $x \in (S_{\lambda_n}^{f_n})_\rho$, then for each $n \in N$, $f_n(x) \leq \lambda_n$ *and* $\|x\| \leq \rho$, hence $(x\ , \lambda_n) \in (epi\, f_n)_{\rho_1}$ with $\rho_1 = Max\left\{\rho\, , |\lambda_n|\right\}$.

By definition (5) of H_{ρ_1}, for each $\varepsilon > 0$, there existm $(\xi_\varepsilon\, , \lambda_\varepsilon) \in epif$ such that, $\|x - \xi_\varepsilon\| \leq H_{\rho_1}(f_n, f) + \varepsilon$

$$|\lambda_n - \lambda_\varepsilon| \leq H_{\rho_1}(f_n, f) + \varepsilon$$

Therefore, we derive that

$$f(\xi_\varepsilon) \leq \lambda_n + H_{\rho_1}(f_n, f) + \varepsilon$$

Since $\lambda_n \xrightarrow[n\to\infty]{} \lambda$ and by the assumption $\lim_{n\to\infty} H_\rho(f_n, f) = 0$, and taking $(\varepsilon \to 0)$, we obtain $f(\xi_\varepsilon) \leq \lambda$, that is $\xi_\varepsilon \in S_\lambda^f$.

Hence for each $x \in (S_{\lambda_n}^{f_n})_\rho ; n \in N$, $d(x, S_\lambda^f) \leq \varepsilon$ That is proving that $e\left[(S_{\lambda_n}^{f_n})_\rho\, , S_\lambda^f\right] \xrightarrow[n\to\infty]{} 0$.

Remark

The condition $\lambda > \inf f$ in theorem 2.1 is necessary for the convergence of the level sets with respect to the $\rho - Housdoroff$ distance. Taking $X = R$ and define the function f by

$$f(x) = \begin{cases} -x & ; x < 0 \\ 0 & ; x \geq 0 \end{cases}$$

For each $n \in N$ we define f_n by

$$f_n(x) = \begin{cases} -x & ; x < 0 \\ \dfrac{x}{n} & ; x \geq 0 \end{cases}$$

Clearly $h_\rho(f_n, f) = haus_\rho(epi\, f_n\, , epi\, f) = \dfrac{\rho}{n}$, hence for each $\rho \geq 0$; we have

$$\lim_{n\to\infty} h_\rho(f_n, f) = 0$$

On the other hand, for $\lambda = \inf f = 0$ we have $S_0^f = R^+$ and $S_0^{f_n} = 0$, hence $\lim_{n \to \infty} haus_\rho (S_{\lambda_n}^{f^n}, S_\lambda^f) \neq 0$.

Proposition 2.3

Let $\{ f_n , f ; n \in N \}$ be a sequence of functions in $\overline{R}^X$. For any $\rho \geq 0$ and every $\varepsilon > 0$, we have

1) If $\lim_{n \to \infty} H_\rho (f_n \; f) = 0$, then $\lim_{n\to\infty} haus_\rho (\varepsilon - \arg\min f_n , \varepsilon - \arg\min f) = 0$

2) If $\inf f_n > \inf f$, and $\lim_{n \to \infty} haus_\rho (\varepsilon - \arg\min f_n , \varepsilon - \arg\min f) = 0$, then

$$\lim_{n \to \infty} H_\rho (f_n ,f) = 0 .$$

Proof :

By definition (2) of $\varepsilon - \arg\min$, we have

$$\varepsilon - \arg\min f_n = \left\{ x \in X \;/ f_n(x) \leq \max \left\{ \inf f_n + \varepsilon , -\frac{1}{\varepsilon} \right\} \right\} := S_{\lambda_n}^{f_n}$$

$$\varepsilon - \arg\min f = \left\{ x \in X \;/ f(x) \leq \max \left\{ \inf f + \varepsilon , -\frac{1}{\varepsilon} \right\} \right\} := S_\lambda^f$$

where

$$\lambda_n = \max \left\{ \inf f_n + \varepsilon , -\frac{1}{\varepsilon} \right\}$$

$$\lambda = \max \left\{ \inf f + \varepsilon , -\frac{1}{\varepsilon} \right\}$$

We noted that $\lambda > \inf f$ and by the proposition 2.2, it is sufficient to proved that $\lambda_n \xrightarrow[n\to\infty]{} \lambda$:

For each $\varepsilon > 0$ $(0 < \varepsilon < 1)$ and each $\rho \geq 0$, let us take $x \in (\varepsilon - \arg\min f_n)_\rho$, then for each $n \in N$,

$$f_n(x) \leq \inf f_n + \varepsilon \quad \textit{and} \quad \| x \| \leq \rho$$

Hence , $(x , \inf f_n + \varepsilon) \in (epi\ f_n)_{\rho_1}$, with

$$\rho_1 = Max\{ \rho , |\inf f_n| + 1 ; |\inf f| + 1\}$$

By definition (5) of H_ρ, for each $\mu > 0$, there exist $(\xi_\mu , \lambda_\mu) \in epif$ such that,

$$\| x - \xi_\mu \| \leq H_{\rho_1}(f_n ,f) + \mu \qquad (16)$$

$$|\inf f_n + \varepsilon - \lambda_\mu| \leq H_{\rho_1}(f_n ,f) + \mu \qquad (17)$$

From (17) it follows that;

$$\lambda_\mu \leq \inf f_n + \varepsilon + H_{\rho_1}(f_n ,f) + \mu$$

and since $f(\xi_\mu) \leq \lambda_\mu$, we derive

$$\inf f \leq f(\xi_\mu) \leq \inf f_n + \varepsilon + H_{\rho_1}(f_n ,f) + \mu$$

Hence

$$\inf f - \inf f_n \leq \varepsilon + H_{\rho_1}(f_n ,f) + \mu$$

Since this inequality is verified for each $\mu > 0$ and each $\varepsilon > 0$, we conclude by taking $(\varepsilon \to 0)$ and $(\mu \to 0)$ that

$$\inf f - \inf f_n \leq H_{\rho_1}(f_n, f)$$

By exchanging the role of f , f_n for each $n \in N$ we obtain

$$\left|\inf f - \inf f_n\right| \leq H_{\rho_1}(f_n, f)$$

and since $H_{\rho_1}(f_n, f) \xrightarrow[n\to\infty]{} 0$, we have $\inf f_n \xrightarrow[n\to\infty]{} \inf f$.

Proposition 2.4

Let X be a Banach space and let $\{ f_n , f ; n \in N \}$ be a sequence of functions in $\Gamma(X)$ such that for each $\rho \geq 0$

$$\lim_{n\to\infty} H_\rho(f_n, f) = 0 .$$

Then for every $\varepsilon > 0$ and each $x \in X$

$$\lim_{n\to\infty} haus_\rho(\partial_\varepsilon f_n(x), \partial_\varepsilon f(x)) = 0 .$$

Proof :

Let $x \in X$, by definition (3) of $\partial_\varepsilon f(x)$, we have

$$\partial_\varepsilon f(x) = \left\{ x^* \in X^* / f(y) \geq f(x) + < y - x, x^* > - \varepsilon ; \forall y \in X \right\}$$

$$= \left\{ x^* \in X^* / f(x) + f^*(x^*) - < x, x^* > \leq \varepsilon \right\}$$

Since $f \in \Gamma(X)$ and X is a Banach space , we have $f^{**} = f$ [15]; Hence

$$\partial_\varepsilon f(x) = \left\{ x^* \in X^* / f^{**}(x) + f^*(x^*) - < x, x^* > \leq \varepsilon \right\}$$

By the definition of f^* we obtain

$$\partial_\varepsilon f(x) = \left\{ x^* \in X^* / f^*(x^*) - < x, x^* > \leq f^*(\xi^*) - < x, \xi^* > + \varepsilon , \forall \xi^* \in X^* \right\}\}$$

$$= \varepsilon - \arg\min F$$

We prove also that $\partial_\varepsilon f_n(x) = \varepsilon - \arg\min F_n$, where

$$F = f^* - < . , x >$$

$$F_n = f_n^* - < . , x >$$

In order to apply (1) to proposition 2.3 , it remains to show that

$$\lim_{n\to\infty} H_\rho(F_n, F) = 0 ; \forall \rho \geq 0$$

In fact , since $\lim_{n\to\infty} H_\rho(f_n, f) = 0$ and from the proposition 1.1,

we have $\lim_{n\to\infty} H_\rho(f_n^*, f^*) = 0 , \forall \rho \geq 0$

and since $x \to <., x>$ is linear continuous function on dom f ,we apply proposition 1.2, to get

$$H_\rho(f_n^* + <., x>, f^* + <., x>) \xrightarrow[n\to\infty]{} 0 ; \forall \rho \geq 0 .$$

This proves that: $\lim_{n\to\infty} H_\rho(F_n, F) = 0$.

Proposition 2.5

Let X be a Banach space and let $\{C_n , C ; n \in N\}$ be a sequence of convex sets in X, and let $x_0 \in X$ be such that for each $\rho \geq 0$

$$\lim_{n\to\infty} haus_\rho(C_n , C) = 0 .$$

Then for every $\varepsilon > 0$ and each $\rho \geq 0$

$$\lim_{n\to\infty} haus_\rho\left(\varepsilon - proj(x_0, C_n), \varepsilon - proj(x_0, C)\right) = 0 .$$

Proof:

By definition $\varepsilon - proj(x_0, C)$, we have

$$\varepsilon - proj(x_0, c) = \{x \in C / \|x - x_0\| \leq d(x_0, C) + \varepsilon\}$$
$$= \varepsilon - \arg\min f$$

We note also that for each $n \in N$

$$\varepsilon - proj(x_0, C_n) = \varepsilon - \arg\min f_n \text{, where}$$
$$f = \|. - x_0\| + \delta(.,C)$$
$$f_n = \|. - x_0\| + \delta(.,C_n)$$

And $\delta(.,C)$ is the indicator function to set C $(=0 \text{ if } x \in C \text{ and } +\infty \text{ if } x \notin C)$.

By the proposition 2.4, it is sufficient to proved that $\lim_{n\to\infty} H_\rho(f_n, f) = 0 \; ; \; \forall \rho \geq 0$

It is clearly that $haus_\rho(C_n, C) = H_\rho[\delta(.,C_n), \delta(.,C)]$. Hencesince $haus_\rho(C_n, C) \xrightarrow[n\to\infty]{} 0$ thus $H_\rho[\delta(.,C_n), \delta(.,C)] \xrightarrow[n\to\infty]{} 0$.

On the other hand, since $x \to \|. - x\|$ is continuous function on X, we apply proposition 1.2 to get :

$$H_\rho(\delta(.,C_n) + \|. - x_0\|, \delta(.,C) + \|. - x_0\|) \xrightarrow[n\to\infty]{} 0 .$$

This proves that $\lim_{n\to\infty} H_\rho(f_n, f) = 0.$

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